Rotation intrinsic spin coupling—the parallelism description

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Abstract

For the Dirac particle in the rotational system, the rotation induced inertia effect is analogously treated as the modification of the "spin connection" on the Dirac equation in the flat spacetime, which is determined by the equivalent tetrad. From the point of view of parallelism description of spacetime, the obtained torsion axial-vector is just the rotational angular velocity, which is included in the "spin connection". Furthermore the axial-vector spin coupling induced spin precession is just the rotation-spin(1/2) interaction predicted by Mashhoon. Our derivation treatment is straightforward and simplified in the geometrical meaning and physical conception, however the obtained conclusions are consistent with that of the other previous work.

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I. INTRODUCTION

Recently the spin-rotation-gravity coupling has been paid much attention and appeared in the work of many authors who have been mainly interested in the study of wave equations in accelerated systems and gravitational fields [1–8]. Indeed, the coupling under consideration here directly involves wave effects that pertain to the physical foundations of general relativity. It follows that similar rotation-spin coupling effects are expected in a rotating frame of reference [1,9].

The observational consequences of rotation-spin coupling for neutron interferometry in a rotating frame of reference have been explored in connection with the assumptions that underlie the physical interpretation of wave equations in an arbitrary frame of reference [10,11]. In general, the rotation-spin phase shift is smaller than the Sagnac shift [12] by roughly the ratio of the wavelength to the dimension of the interferometer.

A proper theoretical treatment of the inertial properties of a Dirac particle is due to Hehl and Ni [2]. This treatment has been extended in several important directions by a number of investigators [2,4,7,8]. The significance of rotation-spin coupling for atomic physics has been pointed out by Silverman [13].

Moreover, the astrophysical consequences of the helicity flip of massive neutrinos as a consequence of rotation-spin coupling have been investigated by Papini et al. [14]. Furthermore, the influence of the rotation-spin coupling on the magnetic inclination evolution of pulsars has also been investigated [15,16].

The direct evidence for the coupling of intrinsic spin to the rotation of the Earth has recently become available [10]. In fact, according to the natural extension of general relativity under consideration here, every spin-$\frac{1}{2}$ particle in the laboratory has an additional interaction Hamiltonian. As measured by the observer, however, such intrinsic spin must "precess" in a sense opposite to the sense of rotation of the Earth. The Hamiltonian associated with such

motion would be of the form [9]

$$\delta H = -\mathbf{\Omega} \cdot \boldsymbol{\sigma} \ , \tag{1}$$

where $\mathbf{\Omega}$ is the frequency of rotation of the laboratory frame. The existence of such a Hamiltonian would show that intrinsic spin has rotational inertia.

In quantum mechanics, mass and spin characterize the irreducible unitary representations of the inhomogeneous Lorentz group. The inertial properties of mass are well known in classical mechanics through various translational and rotational acceleration effects. It is therefore interesting to consider the inertial properties of spin [1,10].

The aim of the present paper is to discuss the rotation-spin effect in straightforward way, i.e., discussing the inertia effect on the Dirac equation by means of the parallelism description of spacetime, so the article is organized as follows, in Section II, we introduce the teleparallel equivalent description of general relativity (GR), and in Section III, we discuss the Dirac equation in GR and in the framework of parallelism description. In Section IV, we extend our discuss to the rotational system, where the torsion axial-vector induced spin precession interaction are studied. The conclusions and further expectation of the teleparallel equivalent description of the axisymmetrical spacetime will appear in Section V. We use the unit in which the speed of light is set equal to unit: $c = 1$.

II. THE TELEPARALLEL EQUIVALENT OF GENERAL RELATIVITY

The teleparallel equivalent of general relativity (PGR) has been pursued by a number of authors [17–19], where the spacetime is characterized by the torsion tensor and the vanishing curvature, the relevant spacetime is the Weitzenböck spacetime [19], which is a special case of the Riemann-Cartan spacetime with the constructed metric-affine theory of gravitation. As is well known, at least in the absence of spinor fields, the teleparallel gravity is equivalent to general relativity. We will use the greek alphabet (μ, ν, ρ, $\cdots = 1, 2, 3, 4$) to denote tensor indices, that is, indices related to spacetime. The latin alphabet (a, b,

$c, \cdots = 1, 2, 3, 4)$ will be used to denote local Lorentz (or tangent space) indices. Of course, being of the same kind, tensor and local Lorentz indices can be changed into each other with the use of the tetrad $h^a{}_\mu$, which satisfy

$$e^a{}_\mu \, e_a{}^\nu = \delta_\mu{}^\nu \quad ; \quad e^a{}_\mu \, e_b{}^\mu = \delta^a{}_b \, . \tag{2}$$

A nontrivial tetrad field can be used to define the linear Cartan connection

$$\Gamma^\sigma{}_{\mu\nu} = e_a{}^\sigma \partial_\nu e^a{}_\mu \, , \tag{3}$$

with respect to which the tetrad is parallel:

$$\nabla_\nu \, e^a{}_\mu \equiv \partial_\nu e^a{}_\mu - \Gamma^\rho{}_{\mu\nu} \, e^a{}_\rho = 0 \, . \tag{4}$$

The Cartan connection can be decomposed according to

$$\Gamma^\sigma{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^\sigma{}_{\mu\nu} + K^\sigma{}_{\mu\nu} \, , \tag{5}$$

where

$$\overset{\circ}{\Gamma}{}^\sigma{}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left[\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right] \tag{6}$$

is the Levi–Civita connection of the metric

$$g_{\mu\nu} = \eta_{ab} \, e^a{}_\mu \, e^b{}_\nu \, , \tag{7}$$

and

$$K^\sigma{}_{\mu\nu} = \frac{1}{2} \left[T_\mu{}^\sigma{}_\nu + T_\nu{}^\sigma{}_\mu - T^\sigma{}_{\mu\nu} \right] \tag{8}$$

is the contorsion tensor, with

$$T^\sigma{}_{\mu\nu} = \Gamma^\sigma{}_{\mu\nu} - \Gamma^\sigma{}_{\nu\mu} \tag{9}$$

the torsion of the Cartan connection [19]. The irreducible torsion vectors, i.e., the torsion vector and the torsion axial-vector, can then be constructed as [19]

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$$V_\mu = T^\nu{}_{\nu\mu} \tag{10}$$

$$A^\mu = \frac{1}{6}\epsilon^{\mu\nu\rho\sigma}T_{\nu\rho\sigma} \tag{11}$$

The nontrivial tetrad field induces both, a riemannian and a teleparallel structures in spacetime. The first is related to the Levi–Civita connection, a connection presenting curvature, but no torsion. The second is related to the Cartan connection, a connection presenting torsion, but no curvature. It is important to remark that both connections are defined on the very same spacetime, a spacetime endowed with both a riemannian and a teleparallel structures.

III. DIRAC EQUATION IN THE CURVED SPACETIME

The gravitational effects on the spin incorporated into Dirac equation through the "spin connection" Γ_μ appearing in the Dirac equation in curved spacetime [20], which is constructed by means of the variation of the covariant Lagrangian of the spinor field as,

$$\left[\gamma^a e_a^\mu(\partial_\mu + \Gamma_\mu) + m\right]\psi = 0. \tag{12}$$

The explicit expression for Γ_μ can be written in terms of the Dirac matrices and tetrads(see also [21])

$$\Gamma_\mu \equiv \frac{1}{8}[\gamma^b, \gamma^c]e_b{}^\nu e_{c\nu;\mu}. \tag{13}$$

We must first simplify the Dirac matrix product in the spin connection term. It can be shown that

$$\gamma^a[\gamma^b, \gamma^c] = 2\eta^{ab}\gamma^c - 2\eta^{ac}\gamma^b - 2i\epsilon^{dabc}\gamma_5\gamma_d, \tag{14}$$

where η^{ab} is the metric in flat space and ϵ^{abcd} is the (flat space) totally antisymmetric tensor, with $\epsilon^{0123} = +1$. With Eq.(14), the contribution from the spin connection is arranged as

$$\Gamma_\mu \equiv \frac{1}{2}V_\mu - \frac{3i}{4}A_\mu\gamma_5, \tag{15}$$

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which means that Eq.(13) and Eq.(15) are equivalent but just the different mathematical form [19]. Alternatively completed in the parallelism description of the Weitzenbock space-time (see Ref. [19]), Dirac equation can be obtained by the variation method, which is constructed by means of the variation of the covariant Lagrangian of the spinor field. The Dirac Langrange density L_D is regularly given by

$$L_D = \frac{1}{2}e_k^\mu[\psi\gamma^k\nabla_\mu\bar\psi - \nabla_\mu\bar\psi\gamma^k\psi] - m\bar\psi\psi. \tag{16}$$

By taking variation with respect to $\bar\psi$, the Dirac equation in Weitzenböck spacetime is given as described in Eq.(12) with the spin connection in Eq.(15). It is interesting to note that the torsion axial-vector represents the deviation of the axial symmetry from the spherical symmetry [22]. In Weitzenböck spacetime, as well as the general version of torsion gravity, it has been shown by many authors [19,22] that the spin precession of a Dirac particle is intimately related to the torsion axial-vector,

$$\frac{d\mathbf{S}}{dt} = -\frac{3}{2}\boldsymbol{A} \times \mathbf{S} \tag{17}$$

where $\mathbf{S}$ is the spin vector of a Dirac particle, and $\boldsymbol{A}$ is the spacelike part of the torsion axial-vector. Therefore, the corresponding extra Hamiltonian is of the form,

$$\delta H = -\frac{3}{2}\boldsymbol{A} \cdot \boldsymbol{\sigma} \tag{18}$$

IV. THE ROTATION-SPIN EFFECT IN THE PARALLELISM DESCRIPTION

Now we discuss the Dirac equation in the rotational coordinate system, and imagine a rotating disk with the angular velocity Ω in the experimental laboratory system, at where the Dirac particle locates, we set the rotation axis in z-direction. In the rotational coordinate system (t,x,y,z), the tetrad components can be obtained from the following line element (c.f. [2,3])

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

$$= \Delta dt^2 + 2\Omega y dt dx - 2\Omega x dt dy$$

$$- (dx^2 + dy^2 + dz^2) , \tag{19}$$

where $\Delta = 1 - \Omega^2 r^2$ and $r^2 = x^2 + y^2$. In the matrix form, the metric and its inverse are written as,

$$g_{\mu\nu} = \begin{pmatrix} \Delta & \Omega y & -\Omega x & 0 \\ \Omega y & -1 & 0 & 0 \\ -\Omega x & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} , \tag{20}$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & \Omega y & -\Omega x & 0 \\ \Omega y & -1 + \Omega^2 y^2 & -\Omega^2 xy & 0 \\ -\Omega x & -\Omega^2 xy & -1 + \Omega^2 x^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} , \tag{21}$$

$$g = det|g_{\mu\nu}| = -1 . \tag{22}$$

The tetrad can be obtained with the subscript μ denoting the column index (c.f. [2]),

$$e^a{}_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\Omega y & 1 & 0 & 0 \\ \Omega x & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{23}$$

with the inverse $e_a{}^\mu = g^{\mu\nu} e^b{}_\nu \eta_{ab}$

$$e_a{}^\mu = \begin{pmatrix} 1 & \Omega y & -\Omega x & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{24}$$

We can inspect that Eqs.(23) and (24) satisfy the conditions in Eqs.(2) and (7). Or equivalently, the tetrad can be expressed by the dual basis of the differential one-form [2] through choosing a coframe of the rotational coordinate system,

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$$\vartheta^{\hat{0}} = dt \,, \tag{25}$$

$$\vartheta^{\hat{1}} = dx - \Omega y dt \,, \tag{26}$$

$$\vartheta^{\hat{2}} = dy + \Omega x dt \,, \tag{27}$$

$$\vartheta^{\hat{3}} = dz \,, \tag{28}$$

with the obtained metric as

$$ds^2 = \eta_{ab} \, \vartheta^a \otimes \vartheta^b \tag{29}$$

is in agreement with Eq.(19) and that in Ref. [2]. From Eqs.(23) and (24), we can now construct the Cartan connection, whose nonvanishing components are:

$$\Gamma^2{}_{01} = \Omega, \quad \Gamma^1{}_{02} = -\Omega, \tag{30}$$

The corresponding nonvanishing torsion components are:

$$T^2{}_{01} = \Omega, \ \ T^1{}_{02} = -\Omega, \tag{31}$$

The torsion vector and the axial torsion-vector are consequently

$$V_\mu = 0, \ \ \mu = 0, 1, 2, 3 \,, \tag{32}$$

$$A_3 = \frac{2}{3}\Omega \,, \ \ A_k = 0, k = 0, 1, 2 \,. \tag{33}$$

As shown, $A_1 = A_2 = 0$ is on account of the Z-axis symmetry which results in the canceling of the x and y components, and then generally we can write $\boldsymbol{A} = \frac{2}{3}\boldsymbol{\Omega}$ and the corresponding additive Hamiltonian induced by the axial-vector spin coupling in Eq.(18) $\delta H = -\boldsymbol{\Omega} \cdot \boldsymbol{\sigma}$, which is expected in Eq.(1) by Mashhoon. From the spacetime geometry view, the torsion axial-vector represents the deviation from the spherical symmetry [22], i.e., which will disappear in the spherical case (Schwarzschild spacetime for instance) and occurs in the axisymmetry case (Kerr spacetime for instance). Therefore the torsion axial-vector corresponds to a inertia field with respect to Dirac particle, which is now explictly expressed by Eq.(17) that

$$\frac{d\mathbf{S}}{dt} = -\mathbf{\Omega} \times \mathbf{S} \ , \tag{34}$$

which is same as that expected by Mashhoon (c.f. Ref. [9]).

V. DISCUSSIONS AND CONCLUSIONS

The inertia effect on the Dirac particle is studied in this work in the framework of the parallelism description of spacetime. In particular, these results are valid for a neutron and a mass neutrino. Therefore, the rotation-spin coupling, predicted by Mashhoon for a neutron wave, has been derived in an alternative way. We recovered the rotation-spin effect in a straightforward derivation, by means of the parallelism description of spacetime, and the rotation-spin effect can be clearly expressed by the spin precession effect of the irreducible torsion axial-vector, which is constructed by the Cartan connection directly. The "noninertia force" on Dirac particle can be preferably treated as a rotation induced torsion of spacetime. Furthermore the constant axial-vector (angular velocity) means that the "noninertia force" is universally same in any spacetime position. However the geometrical and physical meaning of the latter is simply and clearly shown. In the parallelism description of spacetime, the basic element of spacetime is a tetrad, and the metric is a by-product and constructed by the tetrad [19], however this fact is in priority to connect the Dirac equation because the "spin connection" of Dirac equation is described by the tetrad directly but not by the metric. The "spin connection" can be decomposited into two irreducible torsion vectors, i.e., torsion vector and torsion axial-vector, and the latter represents the axisymmetry. The verification and consistency of our derivation in the the parallelism treatment of the inertia effect on Dirac particle leads us to believe that this description would be equivalently extended into the gravitomagnetic effect on Dirac particle [9], where Kerr spacetime induced spin coupling will be examined.

APPENDIX

Dirac matrix in curved spacetime can be given by the standard Dirac-Pauli matrix in the Lorentz coordinates

$$\alpha \equiv \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \quad \beta \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{35}$$

where $\sigma = (\sigma_x, \ \sigma_y, \ \sigma_z)$ is Pauli matrix.

$$\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{36}$$

$$\gamma_i \equiv \beta\alpha = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma_o \equiv \beta \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{37}$$

The Dirac matrix in the rotational coordinates can be expressed by the standard Dirac-Pauli matrix in local Lorentz coordinates (the subscripts are shown with parentheses),

$$\gamma_o = \gamma_{(o)} - \Omega[y\gamma_{(1)} - x\gamma_{(2)}] \tag{38}$$

$$\gamma_1 = \gamma_{(1)}$$

$$\gamma_2 = \gamma_{(2)}$$

$$\gamma_3 = \gamma_{(3)}$$

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